ASYMPTOTIC AND NUMERICAL SOLUTIONS OF THE ORR-SOMMERFELD EQUATION FOR A THIN LIQUID FILM ON AN INCLINED PLANE

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Abstract. An analysis of the initial behavior of free surface liquid flows on inclined planes is presented. The surface of the liquid, under some conditions, may present long-wave instabilities. These instabilities may evolve to surface-waves, that often appear on thin liquid films. Such knowledge is useful in industry, once liquid films help to remove the heat from solid surfaces, and also reduces the friction between high viscosity fluids and pipe walls, by injecting, close to the wall, a less viscous fluid. The surface-waves instability phenomena are governed by the Orr-Sommerfeld equation and their boundary conditions. In this work we present a long-wave solution through an analytical and numerical approaches for the Orr-Sommerfeld equation based on asymptotic and Galerkin methods. Both methods are compared with previous works for validation. The solution gives the critical conditions in which the liquid film turns unstable, and describes possible features that produce these instabilities. All codes, data and plots were produced in the MATLAB environment.

Keywords: Surface waves, Orr-Sommerfeld equation, Asymptotic method, Chebyshev polynomials, Galerkin method.

1. Introduction

The surface of the liquid, under some conditions, may present long-wave instabilities. These instabilities may evolve to surface-waves, that often appear on thin liquid films. Such knowledge is useful in industry, once liquid films help to remove the heat from solid surfaces, and also reduces the friction between high viscosity fluids and pipe walls, by injecting, close to the wall, a less viscous fluid. On the other hand, sometimes we can use instabilities to increase heat and mass transfers, for example, in a reactor cooling process, and they can be provoked, or intensified, by other boundary conditions such as a presence of an electric field (Gonzalez and Castellanos, 1996). An analysis of the initial behavior of free surface liquid flows on inclined planes is presented. The surface-wave instability is governed by the Orr-Sommerfeld equation (Orr, 1907; Sommerfeld, 1908) and their boundary conditions. As boundary conditions of the problem, it was considered the no-slip condition at the wall, kinematic condition of the interface and dynamic condition of the interface. To obtain the Orr-Sommerfeld equation, it is necessary to introduce perturbations in the form of a stream function in the Navier-Stokes equation, as done by Yih (1963), then use normal modes for the stream functions. Similar approach for the development of the Orr-Sommerfeld equation can be found in the works of Schlichting (1979) and Panton (2013). After a linearization of the perturbations, we obtain the equation of Orr-Sommerfeld. For the final form of the boundary conditions the same procedure is applied. For these equations a long-wave solution was considered, based on asymptotic analysis (Kevorkian and Cole, 1981), similar to the approach employed by Smith (1990). For the long-wave perturbations, the wave number can be treated as a small parameter, and the form of the equations suggests that the speed and amplitude of the eigenfunction can be sought as a power series of the wave number. With the asymptotic solution it was possible to find a critical condition for the growth rate of the instability which was expressed in terms of the dimensionless groups Reynolds, Froude and Weber. The numerical solution was based on a Galerkin method (Fletcher, 1984) using Chebyshev polynomials (Boyd, 1989) for the discretization, which made it possible to express the Orr-Sommerfeld equation and their boundary conditions as a generalized eigenvalue problem. All those choices were made because of the general approach provided by the Galerkin method, which makes the implementation of the boundary condition of free surface easier, and the high accuracy of the Chebyshev polynomials. A code was implemented in Matlab to solve the linear system using a QZ algorithm. The main goal of this work is to reach the critical conditions in which the liquid film flow turns unstable in function of the dimensionless groups. Once in possession of this result, it is possible to find the growth rate, the wave speed and the critical Reynolds, Froude and Weber numbers of the liquid film at the instability threshold. The results are compared with previously published data.

2. Formulation of the problem and base state

As references of the development of the momentum equations was considered the work of Fay (1994). The authors considered a film of incompressible Newtonian liquid of viscosity $\mu$, density $\rho$, and thickness $h$, falling down on a inclined...
plane with an angle \( \theta \) with respect to the horizontal and without presence of a gradient temperature. The interface between the liquid and the gas has a surface tension \( \gamma \) and no surfactants is present. The pressure applied by the gas on the interface is \( P_0 \). Figure 1 presents a sketch of the physical problem. For the base state the interface \( H(x, y, t) \), defined in Eq. (1), and \( \eta(x, t) \), defined in Eq. (2), are both planar. Based on the Navier-Stokes equations and considering the boundary conditions of permanent flow, non-slip at solid surface, no shear and constant pressure at the liquid-gas interface, we can find a solution corresponding to a steady parallel flow with a planar interface and parabolic velocity profile:

\[
H(x, y, t) = y - \eta(x, t) \quad (1)
\]

\[
\eta(x, t) = 0 \quad (2)
\]

\[
U(y) = U_0(1 - \frac{y^2}{h^2}) \quad (3)
\]

\[
V(y) = 0 \quad (4)
\]

\[
P(y) = P_0 - \rho g \cos(\theta) y \quad (5)
\]

where \( \eta(x, t) \) is the position of the interface, in function of \( x \) and \( t \), and the velocity of the interface \( U_0 \) is given by,

\[
U_0 = \frac{\rho gh^2 \sin(\theta)}{2\mu} \quad (6)
\]

Figure 1. Liquid film falling down on a inclined plane with parabolic velocity profile, planar interface \( H \), where \( \vec{n} \) is parallel to the gradient of \( H (\nabla H) \), and perturbed interfacial position \( \eta(x, t) \) (Chimetta and Franklin, 2015).

The contribution of inertia relative to viscosity, gravity, and surface tension are measured, respectively, by the Reynolds, Froude, and Weber numbers. The solutions are given in terms of the parameters \( h, \rho \) and \( U_0 \). The dimensionless groups are defined as:

\[
Re = \frac{\rho U_0 h}{\mu}, \quad Fr = \frac{U_0^2}{gh\cos(\theta)} = \frac{Re \tan(\theta)}{2}, \quad We = \frac{\rho U_0^2 h}{\gamma} \quad (7)
\]

The Froude number is defined using the gravity component \( g\cos(\theta) \) normal to the flow and \( \gamma \) is the surface tension. When the interface is perturbed (\( \eta(x, t) \neq 0 \)), the velocity profile no longer has an exact parabolic behaviour. This, combined with inertia, leads to surface waves instabilities.

3. Orr-Sommerfeld equation and boundary conditions

The Squire’s theorem (Squire, 1933) sets that the first instability on parallel flow of a Newtonian fluid is always two-dimensional, and this result remains valid for a flow with interface (Hesla et al., 1986). Considering only the two-dimensional case of the problem, the Navier-Stokes equations and continuity are:

\[
\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \rho g_x
\]

\[
\rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right] = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho g_y
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]
\[ \rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right] = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \rho g_y \]  

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  

Applying the perturbations in the base state we have,

\[ u = U(y) + \hat{u} = U + \frac{\partial \Psi}{\partial y} \]  

\[ v = V(y) + \hat{v} = 0 + -\frac{\partial \Psi}{\partial x} \]  

\[ p = P(y) + \hat{p} \]  

\[ \Xi = \eta(x,t) + \hat{\eta} \]  

where \( \hat{u} = \frac{\partial \Psi}{\partial y}, \hat{v} = -\frac{\partial \Psi}{\partial x} \). Considering only first order perturbations and using a cross differentiation between the equations of motion for \( x \) and \( y \) components we have, in dimensionless form, the equation:

\[ \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 \Psi - \frac{\partial^2 U}{\partial y^2} \frac{\partial \Psi}{\partial x} = \frac{1}{Re} \left[ \nabla^2 (\nabla^2 \Psi) \right] \]  

Considering the normal mode for the stream functions in the form,

\[ \Psi(x,y,t) = \hat{\Psi}(y) e^{i\alpha(x-ct)} \]  

where \( \alpha = \frac{\omega}{k} \in C, \alpha = kh \in R, k \) is the wave number, therefore \( \alpha \) is the characteristic wave number of the problem, and \( \omega \) is the frequency. Using \( \Psi(x,y,t) \) in Eq. (15) and making the contraction of notation for \( D = \frac{\partial}{\partial y} \), we obtain the Orr-Sommerfeld Equation in dimensionless form, given by:

\[ (D^2 - \alpha^2)^2 \hat{\Psi}(y) = i\alpha Re[(\nabla - c)(D^2 - \alpha^2) - D^2 U] \hat{\Psi}(y) \]  

For the boundary conditions of the problem, which are, wall condition, kinematic condition of the interface and dynamic condition of the interface, the same procedure used for Orr-Sommerfeld equation is used here. The wall condition is given by:

\[ u = 0, y = -h \]  

\[ v = 0, y = -h \]  

\[ D \hat{\Psi}(-1) = 0 \]  

\[ \hat{\Psi}(-1) = 0 \]  

Figure 1 presents the interface \( H(x,y,t) \), and the corresponding vectors. The unit vectors normal and tangent to the interface, after a linearisation, are given by:

\[ \vec{n} = \frac{\partial \eta(x,t)}{\partial x} \hat{e}_x + \hat{e}_y \]  

\[ \vec{t} = \hat{e}_x + \frac{\partial \eta(x,t)}{\partial x} \hat{e}_y \]  

where \( \frac{\partial \eta(x,t)}{\partial x} \) is the inclination of the interface. The kinematic condition is,

\[ \vec{u} \cdot \vec{n} = \vec{w} \cdot \vec{n} \text{ for } y = \eta(x,t) \]  

where,

\[ \vec{u} \cdot \vec{n} = -u \frac{\partial \eta(x,t)}{\partial x} + v \]
\[ \vec{w} \cdot \vec{n} = \frac{\partial \eta(x, t)}{\partial t} \]  

(26)

Considering the normal mode for the interface position \( \eta(x, t) \) as,

\[ \eta(x, t) = \hat{\eta} e^{i\alpha(x - ct)} \]  

(27)

where \( \hat{\eta} \) is the amplitude of deformation of the interface. Replacing Eq. (25), Eq. (26) in Eq. (24), using Eq. (16) and Eq. (27), and linearizing the equation around \( y = 0 \),

\[ \hat{\Psi}(0) - (c - 1)\hat{\eta} = 0 \]  

(28)

The dynamic condition at the interface has two parts. The first condition is the continuity of the tangential stress, associated with the viscous stress of the fluid. The second one is the continuity of the normal stress associated with the surface tension. The stress in the fluid is given by \( \Sigma \cdot \vec{n} \), where \( \Sigma \) is the stress tensor, and reducing the effect of air to a purely normal stress \( -P_0 \vec{n} \), the dynamic conditions will be given by:

\[ \vec{t} \cdot (\Sigma \cdot \vec{n}) = 0 \]  

(29)

\[ P_1 - P_2 = -\gamma(\nabla \cdot \vec{n}) \Rightarrow \vec{n} \cdot (\Sigma \cdot \vec{n}) - \vec{n} \cdot (-P_0 \vec{n}) = -\gamma(\nabla \cdot \vec{n}) ; y = \eta(x, t) \]  

(30)

Inserting the normal modes in Eqs. (29) and (30) and linearizing around \( y = 0 \), we obtain:

\[ D^2 \hat{\Psi}(0) + \alpha^2 \hat{\Psi}(0) + \hat{\eta} D^2 \hat{U}(0) = 0 \]  

(31)

\[ -D^3 \hat{\Psi}(0) + [3\alpha^2 - i\alpha Re(c - 1)] D \hat{\Psi}(0) + i\alpha Re \left[ \frac{1}{Fr} + \frac{\alpha^2}{We} \right] \hat{\eta} = 0 \]  

(32)

4. Asymptotic analysis

This section is devoted to the solutions of the equation of Orr-Sommerfeld, together with the boundary conditions of the problem, by means of an asymptotic analysis considered by Chimetta and Franklin (2015). To find these solutions the authors expanded the eigenfunction \( \hat{\Psi}(y) \) and the eigenvalue \( c \) in power series of \( \alpha \), from \( O(1) \) to \( O(\alpha^2) \). For a long wave disturbance, the wave number \( \alpha \) can be treated as a small parameter. The equations suggest the speed \( c \) and the amplitude \( \hat{\Psi} \) of eigenfunctions can be treated as a power series of \( \alpha \). These considerations are possible once that the authors are considering a temporal analysis. In order to achieve an approximate solution we will replace the expansions into the Orr-Sommerfeld equation and the terms of the same order will be collected. The very same procedure will be applied on the boundary conditions.

4.1 Solution for \( O(1) \)

At \( O(1) \):

\[ D^4 \hat{\Psi}_0(y) = 0 \]  

(33)

\[ \hat{\Psi}_0(-1) = 0 \]  

(34)

\[ D\hat{\Psi}_0(-1) = 0 \]  

(35)

\[ \hat{\Psi}_0(0) - (c_0 - 1)\hat{\eta} = 0 \]  

(36)

\[ D^3\hat{\Psi}_0(0) = 0 \]  

(37)

\[ D^2\hat{\Psi}_0(0) - 2 \frac{\hat{\Psi}_0(0)}{c_0 - 1} = 0 \]  

(38)

From equations (33) to (38),

\[ \hat{\Psi}_0(y) = \hat{\eta}(y + 1)^2 ; c_0 = 2 \]  

(39)
4.2 Solution for $O(\alpha)$

At $O(\alpha)$:

\[ D^4 \hat{\Psi}_1(y) = 4iRe\hat{\eta}y \]
\[ \hat{\Psi}_1(-1) = 0 \]
\[ D\hat{\Psi}_1(-1) = 0 \]
\[ \hat{\Psi}_1(0) = c_1\hat{\eta} \]
\[ D^2 \hat{\Psi}_1(0) = 0 \]
\[ D^3 \hat{\Psi}_1(0) = -2iRe\hat{\eta} + iRe\hat{\eta} \left( \frac{1}{Fr} + \frac{\alpha^2}{We} \right) \]

With these equations we find:

\[ \hat{\Psi}_1(y) = iRe\hat{\eta} \left\{ \frac{y^5}{30} + \left[ -\frac{1}{3} + \frac{1}{6} \left( \frac{1}{Fr} + \frac{\alpha^2}{We} \right) \right] y^3 + \left[ \frac{5}{6} - \frac{1}{2} \left( \frac{1}{Fr} + \frac{\alpha^2}{We} \right) \right] y + \frac{8}{15} \right\} \]
\[ c_1 = iRe \left( \frac{8}{15} \right) \left( 1 - \frac{5}{8} \left( \frac{1}{Fr} + \frac{\alpha^2}{We} \right) \right) \]

4.3 Solution for $O(\alpha^2)$

At $O(\alpha^2)$:

\[ D^4 \hat{\Psi}_2(y) = 4\hat{\eta} - Re^2\hat{\eta} \left\{ -\frac{3}{5} y^5 + \left[ \frac{2}{3} - \frac{2}{3} \left( \frac{1}{Fr} + \frac{\alpha^2}{We} \right) \right] y^3 \right\} \]
\[ \hat{\Psi}_2(-1) = 0 \]
\[ D\hat{\Psi}_2(-1) = 0 \]
\[ \hat{\Psi}_2(0) = \hat{\eta}c_2 \]
\[ D^2 \hat{\Psi}_2(0) = -\hat{\eta} \]
\[ D^3 \hat{\Psi}_2(0) = 6\hat{\eta} + Re^2\hat{\eta} \left[ \frac{121}{30} - \frac{5}{2} \left( \frac{1}{Fr} + \frac{\alpha^2}{We} \right) \right] \]

And it is possible to obtain,

\[ \hat{\Psi}_2(y) = \frac{\hat{\eta}}{6} y^4 - Re^2\hat{\eta} \left\{ -\frac{1}{84} y^9 + \frac{1}{21} \left[ 1 - \left( \frac{1}{Fr} + \frac{\alpha^2}{We} \right) \right] y^7 \right\} \]
\[ c_2 = -2 - \frac{32}{63} Re^2 \left[ 1 - \frac{5}{8} \left( \frac{1}{Fr} + \frac{\alpha^2}{We} \right) \right] \]

where $A$, $B$, $C$ and $D$ are, respectively,

\[ A = 6\hat{\eta} + Re^2\hat{\eta} \left[ \frac{57}{30} - \frac{7}{6} \left( \frac{1}{Fr} + \frac{\alpha^2}{We} \right) \right] \]
\[ B = -\hat{\eta} \]
\[ C = -\frac{10\hat{\eta}}{3} - Re^2\hat{\eta} \left[ \frac{1333}{28} - \frac{89}{3} \left( \frac{1}{Fr} + \frac{\alpha^2}{We} \right) \right] \]
\[ D = \hat{\eta}c_2 \]
5. Numerical solution

For the numerical solution the authors implemented a Galerkin method using Chebyshev polynomials for the discretization, in particular we use the Chebyshev polynomials of the first kind, known as $T_n$. In order to implement the method it is necessary to transfer the problem domain to the interval $[-1; 1]$, this allows the use of the orthogonal properties of the Chebyshev polynomials in this interval. We use the transformation,

$$z = 2y + 1; \text{ for } y \in [-1; 0].$$

(60)

After applying Eq. (60) and rearranging the boundary conditions in order to eliminate the term $\hat{n}$, we can discretize $\hat{\Psi}(z)$ with the approximation,

$$\hat{\Psi}(z) = \sum_{k=0}^{N} a_k T_k(z); \; k \in \{Z \mid k \geq 0\}$$

(61)

in this way, it is possible write the Orr-Sommerfeld equation in terms of the inner products required in the Galerkin method. After this process, we can rewrite the problem as eigenvalue problem in the form,

$$[A]_{N \times N} \hat{\Psi} = \sigma [B]_{N \times N} \hat{\Psi}$$

(62)

where $N$ is the number of Chebyshev polynomials to be used and the matrices $A$ and $B$ can be respectively written as $A = A_r + iA_i$ and $B = B_r + iB_i$. To apply the boundary conditions we use the same approximation by Eq. (61) for $z = -1$ for wall conditions and $z = 1$ for the interface conditions, and then replace the last lines in the Eq. (62) with the transformed boundary conditions. The Galerkin method was chosen because of the complexity of the boundary conditions at the interface once, in the Galerkin method, we can use the boundary conditions as line vectors in the final matrices. The choice of using Chebyshev polynomials was made because of their high accuracy, and their orthogonal properties, which makes the implementation easier. A code was written in the MATLAB environment to solve the Eq. (62). We used the function 'eig' which uses a Cholesky factorization or a generalized Schur decomposition (QZ algorithm) based on the properties of $A$ and $B$. If $A$ and $B$ are symmetric, the standard choice will be the Cholesky factorization, otherwise the software will implement the QZ algorithm. All codes implemented in this work, using the Galerkin formulation, were made using a number of Chebyshev polynomials equal to 80.

6. Discussion and results

As presented in section 4 the eigenvalue $c_0$ is real and independent of wave number; therefore, all disturbances are propagated with the same speed $2U_0$, independent of the wave (non-dispersive). Since the imaginary part of $c_0$ is zero, the growth rate of instability is zero, so there is no instability at $O(1)$. Since the ratio of amplitudes $\hat{n}$ and $\hat{\Psi}_0$ are real, the interface $\eta$ and the stream function $\Psi$ are in phase. The perturbation velocities, $u = \frac{\partial \Psi}{\partial y}$ and $v = -\frac{\partial \Psi}{\partial x}$ are, respectively, in and out of phase with the interface. For smaller orders the flow does not manifest any instability for

$$Fr < Fr_c$$

where $Fr_c = \frac{5}{8}$

(63)

When $Fr > Fr_c$, $\sigma$ is negative for every $\alpha$, and the flow of the liquid film is linearly stable. For $Fr > Fr_c$, perturbations of wave number below $\alpha_c$ will be amplified. We can find $\alpha_c$ by,

$$\sigma = 0 \Leftrightarrow \alpha^2 \left[ \frac{Re}{3} \left( \frac{1}{Fr_c} - \frac{1}{Fr} \right) - \frac{Re}{3We} \alpha^2 \right] = 0$$

(64)

Excluding the case $\alpha^2 = 0$ we have,

$$\alpha^2_c = We \left( \frac{1}{Fr_c} - \frac{1}{Fr} \right) \text{ with } Fr_c = \frac{5}{8}$$

(65)

Perturbations with wave number $\alpha > \alpha_c$ are attenuated due to the combined effect of surface tension and viscosity. The number $Fr_c = \frac{5}{8}$ is the critical Froude number above which the liquid film is unstable. For a limiting case of Froude number to the Eq. (65) we have,

$$\lim_{Fr \to \infty} \alpha_c = \lim_{Fr \to \infty} \sqrt{We \left( \frac{1}{Fr_c} - \frac{1}{Fr} \right)} \to \sqrt{\frac{We}{Fr_c}}$$

(66)
where the Eq. (66) it sets that the result of the unstable band should be limited for a Froude number large enough.

At $O(\alpha^2)$, we found the correction of eigenvalue $c_1 = iRe \frac{\pi}{15} [1 - \frac{2}{3} (\frac{\pi}{7} + \frac{\pi}{12})]$, which affects the growth rate of instability. This correction generates as a result a critical Froude number equal to $\frac{\pi}{7}$, which, from Eq. (7), provides the critical Reynolds number $Re_c = \frac{\pi}{15} \cot(\theta)$; therefore, the same condition discussed at the end of section (3.2) can be applied for the critical Reynolds number. This results agrees with Benney’s results (Benney, 1966) for the $O(\alpha^2)$, which is given by $c_1 = \frac{\gamma}{2} Re (Re - \frac{\pi}{3} \cot(\theta)) = iRe^2 (1 - \frac{5}{3} \frac{\pi}{7})$. Benney disregarded the contributions of Weber number until $O(\alpha^3)$, however the same criteria for the onset of instabilities was obtained. The solution for $O(\alpha^2)$ is given by Eq. (55) which affects the phase velocity. This correction in $c$ only affects the wave speed, the real part of eigenvalue, therefore, at $O(\alpha^2)$, long wavelengths are weakly dispersive (Benney, 1966). The value found by Benney was $c_2 = -2 \frac{\pi}{15} Re (Re - \frac{\pi}{3} \cot(\theta)) = -2 \frac{\pi}{15} Re^2 (1 - \frac{5}{3} \frac{\pi}{7})$ which shows good agreement with Eq. (55).

For the validation of the numerical method the authors used as reference a physical problem considered by Charru (2011). For the implementation was used the same values being $\theta = \pi/3$, $\alpha = 0.01$, $We = 0.0001$, $Re = 1$ and $N = 70$, as result was found the spectrum of the eigenvalues in Fig.(2) with the critical eigenvalue highlighted.

For the growth rate $\sigma(\alpha)$ (see Fig. 3), and the stability diagram $\sigma(Re)$ (see Fig. 6), given by Eq. (63) and Eq. (65) was used the physical properties $\mu = 0.001 N\,s/m^2$, $\rho = 998.2071Kg/m^3$, $g = 10m/s^2$, $\gamma = 0.07275 N/m$ for temperature $T = 20^\circ C$ and 0.1 mm thickness, as reference in these plots. For the surface tension $\gamma$ we use the work of Vargaftik et al. (1983) as a reference. In Fig. 3 it was used $\frac{\pi}{7} < \theta < \frac{\pi}{6}$ in order to obtain the $Fr < Fr_c$ and $Fr > Fr_c$ with the asymptotic solution. Figure 4 was made using the numerical approach, for the same range of $\theta$, showing that asymptotic and numerical solution matches with high accuracy for the growth rate. Figure 5 was made with the numerical data and presents the stability diagram with the marginal stability curve ($\sigma = 0$, and $0 < \theta < \frac{\pi}{6}$), which separates the stable and unstable domains. These domains are represent by lines corresponding to negative and positive values for the growth rate, the negative (at left and above the zero curve) and positive (at right and below the zero curve) values corresponding to stable and unstable regions, respectively. This diagram shows that the width of the unstable band tends to zero at the threshold $Fr = Fr_c$ and also that we have different behaviours for the growth rate according with the range of $\alpha$. For the interval $0.04 < \alpha < 0.05$ we can see that the values for the growth rate present in the lines increase “faster” compared to other intervals, implying that this range for the wavenumber is more affected by an increment in the wall slope. Figure 6 shows a comparison between the asymptotic and numerical solutions for the marginal stability curve. Both results are in good agreement especially for $Fr \leq 1$. For $Fr > 1$ both lines separates, however, Eq. (66) predicts a limit for the wave number, namely, $\alpha_c = 0.0705$. Both solutions are in good agreement with this limit so, even for a higher Froude number, a small error will be present between both results.

In Fig. 7 we have the stability diagram for the Reynolds number, where it was considered $\alpha = 0.01$, $\theta = \pi/3$, $We = 0.0001$, and we found $Re_c = 1.9262$ for this new system. As it is possible to observe, this diagram do not presents an asymptotic behaviour. In Fig. 8 we have the stability diagram for Weber number for $\alpha = 0.01$, $\theta = \pi/3$, $Re = 1$, and it was found $We_c = 0.0002248$. Both figures were produced with the data from the numerical solution of the equation Eq. (62) increasing the Reynolds and Weber numbers on each iteration.

7. Conclusion

We found the celerity and the growth rate of instability and also a correction for the celerity by performing the analysis at $O(1)$ up to $O(\alpha^2)$. These results are in good agreement with Benney (1966). With the correction $c_1$ at $O(\alpha)$, presented in Eq. (47), it was possible to find the critical Froude number, which determines a critical condition between the effects of inertia and gravity: when $Fr < \frac{\pi}{7}$ the liquid film is stable, and when $Fr > \frac{\pi}{7}$ the inertial effects dominates the flow and the liquid film becomes unstable, as it is possible to see in Fig. 3 based on Eq. (63). As was presented in the stability diagram of the Fig. 7 for the Reynolds number, if the inertial effects increase the unstable band tends to increase. For surface tension effects, it was shown in Fig. 8, that they are smaller in comparison with inertia and affect higher values of the wave-number ($\alpha > 0.05$). In possession of the critical Froude number we developed an expression, namely Eq. (65), for the marginal stability diagram presented in Fig. 6 together with the numerical data from Eq. (62). With the Galerkin method it was possible to validate the asymptotic solution and find others structures for the growth rate besides the neutral curve ($\sigma = 0$) in the instability diagram, as was shown in Fig. 5. This method shows itself as a good alternative to implement complex boundary conditions in a easier way in order to solve stability problems.

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Figure 2. Spectrum of the eigenvalues with $N = 70$.

Figure 3. Behavior of the growth rate $\sigma(\alpha)$ for $Fr < Fr_c$ and $Fr > Fr_c$ with the asymptotic solution. The dotted line represents the growth rate for the critical Froude number.
Figure 4. Behavior of the growth rate $\sigma(\alpha)$ for $Fr < Fr_c$ and $Fr > Fr_c$ with the numerical data. The dotted line represents the growth rate for the critical Froude number.

Figure 5. Stability diagram for Froude number with $0 < Fr < 10$. Data obtained with the numerical solution of the Eq. (62).
Figure 6. Comparison between the stability diagram for asymptotic and numerical solution. The continuous line represents Eq. (65) for the asymptotic solution and the dotted line is the numerical data.

Figure 7. Stability diagram for Reynolds number with $1 < Re < 100$. Data produced with the numerical solution of the Eq. (62).
Figure 8. Stability diagram for Weber number with $0.0001 < Re < 0.1$. Data generated by the numerical solution of the Eq. (62).

9. References


10. Responsibility notice

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